

ON A RESULT OF GREEN AND GRIFFITHS

QIZHENG YIN

ABSTRACT. We present a simple proof of a result of Green and Griffiths, which states that for the generic curve C of genus $g \geq 4$, the 0-cycle $K \times K - (2g - 2)K_\Delta$ is non-torsion in $\mathrm{CH}^2(C \times C)$. The proof is elementary and works in all characteristics.

Introduction

Let C be a smooth algebraic curve of genus g over a field k . Denote by $K \in \mathrm{CH}^1(C)$ the (class of a) canonical divisor of C . We consider the 0-cycle

$$Z := K \times K - (2g - 2)K_\Delta \in \mathrm{CH}^2(C \times C), \quad (1)$$

where K_Δ is the divisor K on the diagonal $\Delta \subset C \times C$. This cycle is of degree 0 and lies in the kernel of the Albanese map.

It is easy to see that $Z = 0$ when the genus $g = 0, 1, 2$. Faber and Pandharipande showed that also $Z = 0$ when $g = 3$, using the fact that curves of genus 3 are either hyper-elliptic or plane curves. They asked if Z vanishes in general. In [GG03], Green and Griffiths answered this with the following result.

Theorem. — If C is the generic curve of genus $g \geq 4$, then $Z \neq 0$ in $\mathrm{CH}^2(C \times C)_{\mathbb{Q}}$.

Their proof is Hodge-theoretic. It involves lengthy calculations of a delicate infinitesimal invariant. In this note we give an elementary proof of this result, which also works in positive characteristic. It consists of two separate steps:

Step 1: a problem on the Jacobian. — We observe that Z is symmetric, so it naturally lives on the second symmetric power $C^{[2]}$ of C . The latter is closely related to the Jacobian J of C via the map $C^{[2]} \rightarrow J$ (with respect to a point $x_0 \in C$). We show that Z is the pull-back of an explicit codimension 2 cycle W on J , which is tautological in the sense of Polishchuk [Pol07]. In particular, the \mathfrak{sl}_2 -action studied in *loc. cit.* gives a unified proof that $Z = 0$ for $g = 3$.

Step 2: a degeneration argument. — We consider W in the relative setting and it lives on the universal Jacobian $\pi: \mathcal{J} \rightarrow \mathcal{M}_{g,1}$. Although Abel-Jacobi trivial fiberwise, the cycle W gives a class $\mathrm{cl}(W)$ in $H^2(\mathcal{M}_{g,1}, R^2\pi_*\mathbb{Q})$ (over \mathbb{C} , or $H^2(\mathcal{M}_{g,1}, R^2\pi_*\mathbb{Q}_\ell(2))$ in general). If W is trivial on the generic fiber, then there should exist an open subset $U \subset \mathcal{M}_{g,1}$ such that the restriction of $\mathrm{cl}(W)$ is zero in $H^2(U, R^2\pi_*\mathbb{Q})$. So it remains to show that such a U does not exist. A key lemma by Fakhruddin (*cf.* [Fak96], Lemma 4.1) reduces this to an argument on the boundary of $\mathcal{M}_{g,1}$. There we construct explicit families of stable curves and study the cycle class of W . It turns out that even the simplest families of ‘test curves’ will suffice for the proof.

Philosophical note. — It is in general very difficult to detect non-trivial cycles in the kernel of the Abel-Jacobi map. Results in this direction are mostly variational, often obtained by calculating infinitesimal invariants on the generic fiber. The invariants are, essentially, Hodge-theoretic objects associated to certain Leray filtrations, and the calculation is usually difficult.

Now since we work with an Abelian scheme \mathcal{J} , the classical Leray filtration is actually a multiplicative decomposition (*cf.* [Voi12], 4.3.3). Its compatibility with the Beauville decomposition in $\mathrm{CH}(\mathcal{J})_{\mathbb{Q}}$

Date: February 27, 2013.

2010 Mathematics Subject Classification. 14C25, 14H10, 14H40.

Key words and phrases. Curves, Jacobian varieties, algebraic cycles.

tells us exactly in which cohomology group lies the cycle class. Finally via the degeneration argument we take full advantage of the boundary of \mathcal{M}_g (or $\mathcal{M}_{g,1}$), which is missing in the infinitesimal approach.

Although elementary, this method can also be used to detect cycles that lie deeper in the conjectural Bloch-Beilinson filtration (cf. [Yin13]).

Notation and conventions. — We work over a field k of arbitrary characteristic. Since the main result is a geometric statement (cf. note after Theorem 2.6), we assume k to be algebraically closed. From now on, Chow groups are with \mathbb{Q} -coefficients. By a ‘cycle’ we mean the rational equivalence class of a cycle. The word ‘generic’ is taken in the schematic sense. Over \mathbb{C} (or any uncountable field), the term ‘very general’ is often used, which means outside a countable union of Zariski-closed proper subsets of the base scheme (cf. Corollary 2.13).

Acknowledgements. — Most of the ideas and techniques we use are due to Alexander Polishchuk and Najmuddin Fakhruddin. I am deeply grateful to my advisor Ben Moonen, who is always there to help. Special thanks to Rahul Pandharipande for inviting me to ETH Zürich and for his interest.

1. Connections with the Jacobian

1.1. We briefly review Polishchuk’s work on the tautological ring of a Jacobian J (cf. [Pol07]). Let C be a smooth curve of genus g over k , and let (J, θ) be the Jacobian of C . Recall the Beauville decomposition $\mathrm{CH}^i(J) = \oplus_{j=i-g}^i \mathrm{CH}_{(j)}^i(J)$, with

$$\mathrm{CH}_{(j)}^i(J) := \{ \alpha \in \mathrm{CH}^i(J) : [n]^*(\alpha) = n^{2i-j} \alpha \text{ for all } n \in \mathbb{Z} \}.$$

We identify J with its dual J^t via the canonical principal polarization, and denote by \mathcal{P} the Poincaré line bundle on $J \times J$. There is the Fourier transform $\mathcal{F} : \mathrm{CH}_{(j)}^i(J) \xrightarrow{\sim} \mathrm{CH}_{(j)}^{g-i+j}(J)$ given by $\alpha \mapsto \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\alpha) \cdot \mathrm{ch}(\mathcal{P}))$, where $\mathrm{pr}_1, \mathrm{pr}_2$ are the two projections (cf. [Bea86]).

Choose a base point $x_0 \in C$, and let $\iota : C \hookrightarrow J$ be the embedding given by $x \mapsto \mathcal{O}_C(x - x_0)$. Consider the 1-cycle $[C] := [\iota(C)]$ and its components $[C]_{(j)} \in \mathrm{CH}_{(j)}^{g-1}(J)$. Define

$$\begin{aligned} p_i &:= \mathcal{F}([C]_{(i-1)}) \in \mathrm{CH}_{(i-1)}^i(J), \text{ for } i \geq 1, \\ q_i &:= \mathcal{F}(\theta \cdot [C]_{(i)}) \in \mathrm{CH}_{(i)}^i(J), \text{ for } i \geq 0. \end{aligned}$$

The \mathbb{Q} -subalgebra of $\mathrm{CH}(J)$ generated by $\{p_i\}_{i \geq 1}$ and $\{q_i\}_{i \geq 0}$ is called the tautological ring of J , denoted by $\mathcal{T}(J)$. Polishchuk proved that it is stable under \mathcal{F} and the Pontryagin product ‘*’.

There is a natural \mathfrak{sl}_2 -action on $\mathrm{CH}(J)$ and on $\mathcal{T}(J)$ (here $\mathfrak{sl}_2 = \mathbb{Q} \cdot e + \mathbb{Q} \cdot f + \mathbb{Q} \cdot h$), defined by

$$e(\alpha) := p_1 \cdot \alpha, \quad f(\alpha) := -[C]_{(0)} * \alpha,$$

and

$$h(\alpha) := (2i - j - g)\alpha, \text{ for } \alpha \in \mathrm{CH}_{(j)}^i(J).$$

Polishchuk also showed that f acts on $\mathcal{T}(J)$ via the differential operator \mathcal{D} given by

$$\mathcal{D} := -\frac{1}{2} \sum_{i,j \geq 1} \binom{i+j}{j} p_{i+j-1} \partial_{p_i} \partial_{p_j} - \sum_{i,j \geq 1} \binom{i+j-1}{j} q_{i+j-1} \partial_{q_i} \partial_{p_j} + \sum_{i \geq 1} q_{i-1} \partial_{p_i}. \quad (2)$$

Now consider the map $\phi : C \times C \rightarrow J$ given by $(x, y) \mapsto \mathcal{O}_C(x + y - 2x_0)$. We would like to express the cycle $Z \in \mathrm{CH}^2(C \times C)$ in (1) as the pull-back of a certain cycle $W \in \mathcal{T}^2(J)$ under ϕ . Since Z is Abel-Jacobi trivial, we should look for W in $\mathcal{T}_{(2)}^2(J)$, which is spanned by q_1^2 and q_2 .

1.2. *Proposition.* — We have $Z = \phi^*(W)$ with $W := 2(q_1^2 - (2g-2)q_2) \in \mathcal{T}_2^2(J)$.

Proof. This is done by an explicit calculation. The essential ingredients are the pull-back of θ via $\iota: C \rightarrow J$, and the pull-back of $c_1(\mathcal{P})$ via $(\iota, \phi): C \times (C \times C) \rightarrow J \times J$. Write $\xi := \iota^*(\theta)$ and $\ell := (\iota, \phi)^*(c_1(\mathcal{P}))$. Then we have

$$\xi = \frac{1}{2}K + [x_0] \quad (\text{cf. [Pol07], (1.1)})$$

and

$$\ell = [\Delta_1] + [\Delta_2] - 2\text{pr}_1^*([x_0]) - \text{pr}_2^*([x_0 \times C] + [C \times x_0]), \quad (\text{cf. [Pol07b], (2.1)})$$

where $\Delta_1 = \{(x, x, y) : x, y \in C\}$ and $\Delta_2 = \{(x, y, x) : x, y \in C\}$, and $\text{pr}_1: C \times (C \times C) \rightarrow C$, $\text{pr}_2: C \times (C \times C) \rightarrow C \times C$ are the projections.

By chasing through several cartesian squares, we have

$$\begin{aligned} \phi^*(\mathcal{F}(\theta \cdot [C])) &= \text{pr}_{2,*}(\text{pr}_1^*(\xi) \cdot \exp(\ell)) \\ &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot \exp([\Delta_1] + [\Delta_2] - 2\text{pr}_1^*([x_0]))\right) \cdot \exp(-[x_0 \times C] - [C \times x_0]) \\ &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\left(\frac{1}{2}K + [x_0]\right) \cdot \exp(-2[x_0])\right) \cdot \exp([\Delta_1] + [\Delta_2])\right) \cdot \exp(-[x_0 \times C] - [C \times x_0]) \\ &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot \exp([\Delta_1] + [\Delta_2])\right) \cdot \exp(-[x_0 \times C] - [C \times x_0]). \end{aligned}$$

Then by expanding the exponentials while keeping track of the codimension, we get

$$\begin{aligned} \phi^*(q_1) &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot ([\Delta_1] + [\Delta_2])\right) - \text{pr}_{2,*}\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot ([x_0 \times C] + [C \times x_0]) \\ &= \frac{1}{2}(K \times [C] + [C] \times K) - (g-1)([x_0 \times C] + [C \times x_0]) \end{aligned}$$

and

$$\begin{aligned} \phi^*(q_2) &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot \frac{1}{2}([\Delta_1] + [\Delta_2])^2\right) \\ &\quad - \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot ([\Delta_1] + [\Delta_2])\right) \cdot ([x_0 \times C] + [C \times x_0]) \\ &\quad + \text{pr}_{2,*}\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot \frac{1}{2}([x_0 \times C] + [C \times x_0])^2 \\ &= \text{pr}_{2,*}\left(\text{pr}_1^*\left(\frac{1}{2}K + [x_0]\right) \cdot ([\Delta_1] \cdot [\Delta_2])\right) - \frac{1}{2}(K \times [x_0] + [x_0] \times K) + (g-2)[x_0 \times x_0] \\ &= \frac{1}{2}K_\Delta - \frac{1}{2}(K \times [x_0] + [x_0] \times K) + (g-1)[x_0 \times x_0]. \end{aligned}$$

Hence

$$\phi^*(q_1^2) = \frac{1}{2}K \times K - (g-1)(K \times [x_0] + [x_0] \times K) + 2(g-1)^2[x_0 \times x_0],$$

and we obtain $\phi^*(2(q_1^2 - (2g-2)q_2)) = K \times K - (2g-2)K_\Delta$. \square

1.3. *Corollary.* — (i) We have $Z = 0$ if and only if $W = 0$. In particular, that W vanishes or not is independent of the point $x_0 \in C$.

(ii) If $g = 3$, then $Z = 0$ in $\text{CH}^2(C \times C)$.

Proof. For (i), we know that $W \in \text{CH}(h^2(J))$, where $h^2(J)$ is the second component in the motivic decomposition of $h(J)$ (cf. [DM91]). The map ϕ induces

$$\phi^*: h^2(J) \xrightarrow{\sim} S^2(h^1(C)) \hookrightarrow h^2(C^{[2]}) \hookrightarrow h^2(C \times C),$$

which realizes $h^2(J)$ as a direct summand of $h^2(C \times C)$. Therefore $\phi^*|_{\text{CH}(h^2(J))}$ is injective.

For (ii), consider the cycle $p_2^2 \in \mathcal{T}_{(2)}^4(J)$. When $g = 3$, we have $p_2^2 = 0$ for dimension reasons. Apply twice the differential operator \mathcal{D} in (2), and we obtain

$$\mathcal{D}^2(p_2^2) = \mathcal{D}(-6p_3 + 2q_1p_1) = 2(q_1^2 - 4q_2) = 0.$$

So $W = 0$, and thus $Z = 0$. \square

1.4. *Remarks.* — (i) By a classical theorem of Roïtman [Roï80], the vanishing of Z with \mathbb{Q} -coefficients implies its vanishing with \mathbb{Z} -coefficients.

(ii) It would be interesting to study the vanishing locus of Z when $g \geq 4$. Also conjecturally Z vanishes if the curve C is defined over $\overline{\mathbb{Q}}$.

2. Fakhruddin's degeneration argument

2.1. We switch to the relative setting and we include stable curves of compact type. Let S be a smooth connected variety of dimension d over k . Let $p: \mathcal{C} \rightarrow S$ be a family of stable 1-pointed curves of genus g and of compact type. Denote by $\pi: \mathcal{J} \rightarrow S$ the relative Jacobian of \mathcal{C} , which is an Abelian scheme. We write $x_0: S \rightarrow \mathcal{C}$ for the section (marked point) of \mathcal{C} , and $\sigma_0: S \rightarrow \mathcal{J}$ for the zero section of \mathcal{J} . As in the absolute case, there is an embedding $\iota: \mathcal{C} \hookrightarrow \mathcal{J}$ that gives us a diagram.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\iota} & \mathcal{J} \\ \downarrow p & & \downarrow \pi \\ S & \xrightarrow{\sigma_0} & \mathcal{J} \end{array} \quad \begin{array}{c} \nearrow x_0 \\ \searrow \end{array} \quad (3)$$

2.2. On $\mathrm{CH}(\mathcal{J})$, we again have a decomposition $\mathrm{CH}^i(\mathcal{J}) = \bigoplus \mathrm{CH}_{(j)}^i(\mathcal{J})$ such that $[n]^*$ is the multiplication by n^{2i-j} on $\mathrm{CH}_{(j)}^i(\mathcal{J})$ (cf. [DM91]). Regarding the ℓ -adic realization of the motive $h(\mathcal{J}/S)$, there is a canonical decomposition

$$R\pi_*\mathbb{Q}_\ell(r) \simeq \sum_i R^i\pi_*\mathbb{Q}_\ell(r)[-i], \quad ('(r)' \text{ for Tate twists})$$

with $[n]^*$ acting on $R^i\pi_*\mathbb{Q}_\ell(r)$ by the multiplication by n^i (cf. [Del68], 2.19). This decomposition is compatible with the multiplicative structure

$$R\pi_*\mathbb{Q}_\ell(r) \otimes R\pi_*\mathbb{Q}_\ell(r') \rightarrow R\pi_*\mathbb{Q}_\ell(r+r')$$

given by the cup product (cf. [Voi12], 4.3.3). It follows that we have a multiplicative decomposition

$$H^m(\mathcal{J}, \mathbb{Q}_\ell(r)) \simeq \bigoplus_{i+j=m} H^j(S, R^i\pi_*\mathbb{Q}_\ell(r)).$$

Comparing the action of $[n]$ on Chow groups and on cohomology, we know that the cycle class map $\mathrm{cl}: \mathrm{CH}^i(\mathcal{J}) \rightarrow H^{2i}(\mathcal{J}, \mathbb{Q}_\ell(i))$ decomposes as a sum of maps

$$\mathrm{cl}: \mathrm{CH}_{(j)}^i(\mathcal{J}) \rightarrow H^j(S, R^{2i-j}\pi_*\mathbb{Q}_\ell(i)), \quad (4)$$

which respect the multiplicative structures on both sides. Note that if the base field $k = \mathbb{C}$, one may work with singular cohomology with coefficients in \mathbb{Q} .

2.3. Consider a cycle $\alpha \in \mathrm{CH}_{(j)}^i(\mathcal{J})$. Denote by J_η the generic fiber of $\mathcal{J} \rightarrow S$, and by $\alpha_\eta \in \mathrm{CH}_{(j)}^i(J_\eta)$ the restriction of α to J_η . Now suppose $\alpha_\eta = 0$. By the ‘spreading-out’ procedure (cf. [Voi12], 2.1), there exists a non-empty open subset $U \subset S$ such that $\alpha_U = 0 \in \mathrm{CH}_{(j)}^i(\mathcal{J}_U)$, where $\mathcal{J}_U := \mathcal{J} \times_S U$ and $\alpha_U := \alpha|_{\mathcal{J}_U}$. Combining with the cycle class map (4), we have the following implication.

2.4. *Proposition.* — If $\alpha_\eta = 0$, then there exists a non-empty open subset $U \subset S$ such that

$$\mathrm{cl}(\alpha_U) = 0 \in H^j(U, R^{2i-j}\pi_*\mathbb{Q}_\ell(i)). \quad \square$$

We consider the cycles q_1^2, q_2 and W (cf. Section 1.1) in the relative setting (3). More precisely, denote by $\theta \in \mathrm{CH}_{(0)}^1(\mathcal{J})$ the divisor class corresponding to the canonical principal polarization $\lambda: \mathcal{J} \xrightarrow{\sim} \mathcal{J}^t$ (so 2θ is the pull-back of the first Chern class of the Poincaré bundle \mathcal{P} under the map $\mathrm{id} \times \lambda: \mathcal{J} \rightarrow \mathcal{J} \times \mathcal{J}^t$). Again we identify \mathcal{J} with \mathcal{J}^t and regard the Fourier transform \mathcal{F} as an endomorphism of $\mathrm{CH}(\mathcal{J})$. Generalizing the definitions of Section 1.1, we write $[\mathcal{C}] := [\iota(\mathcal{C})]$ and let

$$q_i := \mathcal{F}(\theta \cdot [\mathcal{C}]_{(i)}) \in \mathrm{CH}_{(i)}^i(\mathcal{J}), \text{ for } i \geq 0.$$

If the context requires it, we shall denote these cycles by $q_i(\mathcal{C})$. As before, we define $W := 2(q_1^2 - (2g-2)q_2) \in \mathrm{CH}_{(2)}^2(\mathcal{J})$.

Our main focus is the case $S = \mathcal{M}_{g,1}^{\mathrm{ct}}$, i.e. the moduli stack of stable 1-pointed curves of genus g and of compact type. The fact that $\mathcal{M}_{g,1}^{\mathrm{ct}}$ is a stack plays no role in the discussion. In fact, since the Chow groups are with \mathbb{Q} -coefficients, for our purpose (cf. Theorem 2.6) it is equivalent to work over a finite cover of the moduli stack that is an honest variety.

The goal is to prove that for $g \geq 4$, we have $W \neq 0$ generically over $\mathcal{M}_{g,1}^{\mathrm{ct}}$. After Proposition 2.4, we would like to show that for all non-empty open subsets $U \subset \mathcal{M}_{g,1}^{\mathrm{ct}}$, we have

$$\mathrm{cl}(W_U) \neq 0 \in H^2(U, R^2\pi_*\mathbb{Q}_\ell(2)).$$

Using the following lemma by Fakhruddin we can reduce the proof of this to a calculation on the boundary of $\mathcal{M}_{g,1}^{\mathrm{ct}}$.

2.5. *Lemma* ([Fak96], Lemma 4.1). — Let X, S be smooth connected varieties over k and $\pi: X \rightarrow S$ be a smooth proper map. Consider a class $h \in H^m(X, \mathbb{Q}_\ell(r))$. Suppose there exists a non-empty subvariety $T \subset S$ such that for all non-empty open subsets $V \subset T$, we have $h_V \neq 0$, where $h_V := h|_{X_V}$. Then for all non-empty open subsets $U \subset S$, we have $h_U \neq 0$. \square

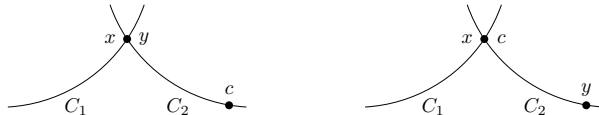
Therefore to achieve the goal, it suffices to construct a family of ‘test curves’ over a variety T on the boundary of $\mathcal{M}_{g,1}^{\mathrm{ct}}$, and to show that the class of W does not vanish over any non-empty open subset of T . In fact, we can prove a slightly stronger result.

2.6. *Theorem.* — When $g \geq 4$, the cycles q_1^2 and q_2 are linearly independent on the generic Jacobian (over $\mathcal{M}_{g,1}^{\mathrm{ct}}$). In particular, we have $W \neq 0$ on the generic Jacobian.

Note that Theorem 2.6 is of geometric nature: if the statement is true over the base field k , then it is automatically true over any base field $k' \subset k$. Therefore the theorem still holds over an arbitrary field (not necessarily algebraically closed). Together with Corollary 1.3 (i), it implies the result of Green and Griffiths.

The rest of this paper is devoted to the construction of the ‘test curves’ and the proof of Theorem 2.6. We shall construct two families of curves over the same base scheme T . We will show that for any non-trivial linear combination of q_1^2 and q_2 , at least one of the two families will give a cohomology class that does not vanish when restricted to non-empty open subsets of T . For simplicity, we begin with the case $g = 4$, while the proof for the general case is almost identical (cf. Section 2.12).

2.7. *Case $g = 4$.* — Take two smooth curves C_1 and C_2 of genus 2 over k , with Jacobians (J_1, θ_1) and (J_2, θ_2) . Let x (resp. y) be a varying point on C_1 (resp. C_2), and c be a fixed point on C_2 . We construct the first family of stable curves by joining x and y and using c as the marked point, and then the second family by joining x and c and using y as the marked point, as is shown in the picture below.



With x and y varying, both families have the same base scheme $T := C_1 \times (C_2 \setminus \{c\})$. We denote them by $\mathcal{C} \rightarrow T$ and $\mathcal{C}' \rightarrow T$ respectively. Observe that \mathcal{C} and \mathcal{C}' have also the same relative Jacobian $\mathcal{J} := J_1 \times J_2 \times T$, a constant Abelian scheme over T via the last projection.

2.8. Consider the embeddings $\mathcal{C} \hookrightarrow \mathcal{J}$ with respect to c , and $\mathcal{C}' \hookrightarrow \mathcal{J}$ with respect to y . An important fact is that both embeddings naturally extend over $C_1 \times C_2 \supset T$. More precisely, we have

$$\begin{aligned} \psi_1: C_1 \times C_1 \times C_2 &\hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } (z, x, y) \mapsto (\mathcal{O}_{C_1}(z - x), \mathcal{O}_{C_2}(y - c), x, y), \\ \psi_2: C_2 \times C_1 \times C_2 &\hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } (w, x, y) \mapsto (0, \mathcal{O}_{C_2}(w - c), x, y), \\ \psi'_1: C_1 \times C_1 \times C_2 &\hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } (z, x, y) \mapsto (\mathcal{O}_{C_1}(z - x), \mathcal{O}_{C_2}(c - y), x, y), \\ \psi'_2: C_2 \times C_1 \times C_2 &\hookrightarrow J_1 \times J_2 \times C_1 \times C_2 \text{ given by } (w, x, y) \mapsto (0, \mathcal{O}_{C_2}(w - y), x, y). \end{aligned}$$

We take $\overline{T} := C_1 \times C_2$ as the base scheme and view the other schemes as \overline{T} -schemes through the projections onto the last two factors. We also write $\overline{\mathcal{J}} := J_1 \times J_2 \times \overline{T}$.

Let $\overline{\mathcal{C}} \subset \overline{\mathcal{J}}$ be the union of the images of ψ_1 and ψ_2 ; similarly, let $\overline{\mathcal{C}'} \subset \overline{\mathcal{J}}$ be the union of the images of ψ'_1 and ψ'_2 . We see that the restriction of $\overline{\mathcal{C}}$ (resp. $\overline{\mathcal{C}'}$) to T is exactly \mathcal{C} (resp. \mathcal{C}'). Write $\theta := \theta_1 \times [J_2] \times [\overline{T}] + [J_1] \times \theta_2 \times [\overline{T}]$, and we have $\theta \in \mathrm{CH}_{(0)}^1(\overline{\mathcal{J}})$. Define

$$\overline{q}_i := \mathcal{F}(\theta \cdot [\overline{\mathcal{C}}]_{(i)}) \in \mathrm{CH}_{(i)}^i(\overline{\mathcal{J}}), \quad \overline{q}'_i := \mathcal{F}(\theta \cdot [\overline{\mathcal{C}'}]_{(i)}) \in \mathrm{CH}_{(i)}^i(\overline{\mathcal{J}}).$$

Again, the restriction of \overline{q}_i (resp. \overline{q}'_i) to T is exactly $q_i(\mathcal{C})$ (resp. $q_i(\mathcal{C}')$).

2.9. As $\overline{\mathcal{J}}$ is a constant Abelian scheme over \overline{T} , we have a Künneth decomposition

$$H^m(\overline{\mathcal{J}}) = \bigoplus_{a_1+b_1+a_2+b_2=m} H^{a_1}(J_1) \otimes H^{b_1}(C_1) \otimes H^{a_2}(J_2) \otimes H^{b_2}(C_2).$$

Here, and in what follows, we omit the coefficients of the cohomology groups. Also on the right hand side we have sorted the factors in the order J_1 - C_1 - J_2 - C_2 , as this turns out to be convenient in our calculations. Given a class $h \in H^m(\overline{\mathcal{J}})$, we denote by $h^{[a_1, b_1, a_2, b_2]}$ its Künneth component in the indicated degrees.

In this case, the cycle class map (4) takes the form

$$\mathrm{cl}: \mathrm{CH}_{(j)}^i(\overline{\mathcal{J}}) \rightarrow \bigoplus_{\substack{a_1+a_2=2i-j \\ b_1+b_2=j}} H^{a_1}(J_1) \otimes H^{b_1}(C_1) \otimes H^{a_2}(J_2) \otimes H^{b_2}(C_2). \quad (5)$$

So for $\alpha = \overline{q}_1^2, \overline{q}_2, \overline{q}_1'^2, \overline{q}_2' \in \mathrm{CH}_{(2)}^2(\overline{\mathcal{J}})$, that $\mathrm{cl}(\alpha)^{[a_1, b_1, a_2, b_2]} \neq 0$ is possible only if $a_1 + a_2 = 2$ and $b_1 + b_2 = 2$. Moreover, we remark that $H^2(C_1)$ (resp. $H^2(C_2)$) is supported on a point of C_1 (resp. C_2). As we should like to have the cycle classes after restriction to open subsets $V \subset T \subset \overline{T}$, the only interesting components are $\mathrm{cl}(\alpha)^{[a_1, 1, a_2, 1]}$ with $a_1 + a_2 = 2$ (in fact, we will see in the proof of Proposition 2.10 that for $\mathrm{cl}(\alpha)^{[a_1, 1, a_2, 1]}$ to be non-zero, we also have $a_1 = a_2 = 1$).

The following elementary calculation is the key point in the proof of Theorem 2.6.

2.10. *Proposition.* — *There exist non-zero classes*

$$\begin{aligned} h_1 &\in H^1(J_1) \otimes H^1(C_1) \otimes H^1(J_2) \otimes H^1(C_2) \\ h_2, h_4 &\in H^0(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2) \\ h_3 &\in H^1(J_1) \otimes H^1(C_1) \otimes H^0(J_2) \otimes H^0(C_2) \end{aligned}$$

such that

$$\begin{aligned} \mathrm{cl}(\overline{q}_2)^{[1, 1, 1, 1]} &= h_1, & \mathrm{cl}(\overline{q}_1^2)^{[1, 1, 1, 1]} &= 2h_2 \smile h_3, \\ \mathrm{cl}(\overline{q}_2')^{[1, 1, 1, 1]} &= -h_1, & \mathrm{cl}(\overline{q}_1'^2)^{[1, 1, 1, 1]} &= -2h_2 \smile h_3 + 2h_3 \smile h_4. \end{aligned}$$

Moreover, the classes $h_2 \smile h_3$ and $h_3 \smile h_4$ are also non-zero.

Proof. The proof is just a careful analysis of the embeddings $\psi_1, \psi_2, \psi'_1, \psi'_2$ defined in Section 2.8. We first calculate the relevant Künneth components of $\text{cl}([\overline{\mathcal{C}}]_{(i)})$ and $\text{cl}([\overline{\mathcal{C}'}]_{(i)})$. Then by intersecting with $\text{cl}(\theta)$ and applying \mathcal{F} in cohomology, we obtain the relevant components of $\text{cl}(\overline{q_i})$ and $\text{cl}(\overline{q'_i})$.

We start with the cycle classes of $[\overline{\mathcal{C}}]_{(1)}$ and $[\overline{\mathcal{C}}]_{(2)}$. Observe that the image of ψ_2 only gives a class in $H^4(J_1) \otimes H^0(C_1) \otimes H^2(J_2) \otimes H^0(C_2)$, which by (5), does not contribute to either $[\overline{\mathcal{C}}]_{(1)}$ or $[\overline{\mathcal{C}}]_{(2)}$. Regarding ψ_1 , we may view it as the product of

$$\begin{aligned} \psi_3: C_1 \times C_1 &\hookrightarrow J_1 \times C_1 & \psi_4: C_2 &\hookrightarrow J_2 \times C_2 \\ (z, x) &\mapsto (\mathcal{O}_{C_1}(z - x), x), & y &\mapsto (\mathcal{O}_{C_2}(y - c), y). \end{aligned}$$

The class of $\text{Im}(\psi_3)$ has components in $H^2(J_1) \otimes H^0(C_1)$, $H^1(J_1) \otimes H^1(C_1)$ and $H^0(J_1) \otimes H^2(C_1)$. The third component is irrelevant due to the appearance of $H^2(C_1)$. We claim that the other two components are both non-zero. To see the first, we regard $J_1 \times C_1$ as a constant family over C_1 . Then $C_1 \times C_1$ is fiberwise an ample divisor, which gives a non-zero class in $H^2(J_1) \otimes H^0(C_1)$. For the component in $H^1(J_1) \otimes H^1(C_1)$, we consider

$$\begin{aligned} C_1 \times C_1 &\xrightarrow{(\text{id}, \Delta)} C_1 \times C_1 \times C_1 \xrightarrow{(\sigma, \text{id})} C_1^{[2]} \times C_1 \xrightarrow{(\varphi, \text{id})} J_1 \times C_1 \\ (z, x) &\mapsto (z, x, x) \mapsto ((z, x), x) \mapsto (\mathcal{O}_{C_1}(z + x - 2x), x). \end{aligned}$$

The class of the diagonal in $C_1 \times C_1$ has a component in $H^1(C_1) \otimes H^1(C_1)$ which, viewed as a correspondence, gives the identity $H^1(C_1) \xrightarrow{\sim} H^1(C_1)$. It follows that the class of $\text{Im}(\text{id}, \Delta)$ has a non-zero component in $H^0(C_1) \otimes H^1(C_1) \otimes H^1(C_1)$. Moreover, we have isomorphisms

$$\sigma_*: H^0(C_1) \otimes H^1(C_1) \xrightarrow{\sim} H^1(C_1^{[2]}), \quad \varphi_*: H^1(C_1^{[2]}) \xrightarrow{\sim} H^1(J_1),$$

the latter due to the fact that $C_1^{[2]}$ is obtained by blowing up a point in J_1 . Therefore $\text{Im}(\psi_3)$ as a correspondence gives an isomorphism $H^1(J_1) \xrightarrow{\sim} H^1(C_1)$, which implies a non-zero component in $H^1(J_1) \otimes H^1(C_1)$.

Similarly, the class of $\text{Im}(\psi_4)$ has non-zero components in $H^4(J_2) \otimes H^0(C_2)$ and $H^3(J_2) \otimes H^1(C_2)$. Now we collect all non-zero contributions to the classes of $[\overline{\mathcal{C}}]_{(1)}$ and $[\overline{\mathcal{C}}]_{(2)}$ that do not involve either $H^2(C_1)$ or $H^2(C_2)$. For $[\overline{\mathcal{C}}]_{(2)}$, there is only one non-zero class

$$h_1^0 \in H^1(J_1) \otimes H^1(C_1) \otimes H^3(J_2) \otimes H^1(C_2).$$

By intersecting with $\text{cl}(\theta)$ and applying \mathcal{F} , we obtain a non-zero class

$$h_1 \in H^1(J_1) \otimes H^1(C_1) \otimes H^1(J_2) \otimes H^1(C_2),$$

For $[\overline{\mathcal{C}}]_{(1)}$, there are two non-zero classes

$$h_2^0 \in H^2(J_1) \otimes H^0(C_1) \otimes H^3(J_2) \otimes H^1(C_2), \quad h_3^0 \in H^1(J_1) \otimes H^1(C_1) \otimes H^4(J_2) \otimes H^0(C_2).$$

Again by intersecting with $\text{cl}(\theta)$ and applying \mathcal{F} , we obtain non-zero classes

$$h_2 \in H^0(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2), \quad h_3 \in H^1(J_1) \otimes H^1(C_1) \otimes H^0(J_2) \otimes H^0(C_2).$$

It follows that $\text{cl}(\overline{q_2})^{[1,1,1,1]} = h_1$ and $\text{cl}(\overline{q_1}^2)^{[1,1,1,1]} = h_2 \smile h_3 + h_3 \smile h_2 = 2h_2 \smile h_3$.

For the cohomology classes of $\overline{q_2}'$ and $\overline{q_1}'^2$, we remark that the embedding ψ'_1 differs from ψ_1 only by an action of $[-1]$ on the J_2 factor. As a consequence, by repeating the same procedure we obtain classes $h'_1 = -h_1$, $h'_2 = -h_2$ and $h'_3 = h_3$, so that $2h'_2 \smile h'_3 = -2h_2 \smile h_3$. However, this time the embedding ψ'_2 makes an additional contribution. The class of $\text{Im}(\psi'_2)$ has a non-zero component

$$h_4^0 \in H^4(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2),$$

which belongs to the class of $[\overline{\mathcal{C}'}]_{(1)}$. By intersecting with $\text{cl}(\theta)$ and applying \mathcal{F} , we get a non-zero class

$$h_4 \in H^0(J_1) \otimes H^0(C_1) \otimes H^1(J_2) \otimes H^1(C_2).$$

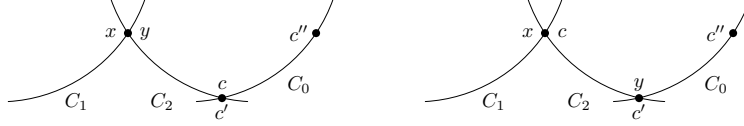
Therefore we have $\text{cl}(\overline{q_2}')^{[1,1,1,1]} = -h_1$ and $\text{cl}(\overline{q_1}'^2)^{[1,1,1,1]} = -2h_2 \smile h_3 + 2h_3 \smile h_4$.

Finally, since the 0-th cohomology groups $H^0(C_i)$ and $H^0(J_i)$ are generated by the unit of the ring structures, we see that both $h_2 \sim h_3$ and $h_3 \sim h_4$ are non-zero. \square

2.11. As $h_1 \neq 0$ and $h_3 \sim h_4 \neq 0$, it follows from Proposition 2.10 that that for any $(r, s) \neq (0, 0) \in \mathbb{Q}^2$, at least one of $\text{cl}(r \cdot \overline{q_1}^2 + s \cdot \overline{q_2})^{[1,1,1,1]}$ and $\text{cl}(r \cdot \overline{q_1'}^2 + s \cdot \overline{q_2'})^{[1,1,1,1]}$ is non-zero in $H^1(C_1) \otimes H^1(J_1) \otimes H^1(C_2) \otimes H^1(J_2)$.

It remains to ensure that this non-zero cohomology class does not vanish when restricted to non-empty open subsets of $\overline{T} = C_1 \times C_2$, i.e. that it is not supported on a divisor of $C_1 \times C_2$. We can achieve this by imposing additional assumptions on C_1 and C_2 . In positive characteristic, we choose C_1 to be ordinary and C_2 supersingular. Over $\overline{\mathbb{Q}}$, and hence for any $k = \overline{k}$ of characteristic 0, we take C_1 and C_2 such that J_1 and J_2 are both simple, and such that $\text{End}(J_1) = \mathbb{Z}$ and J_2 is of CM type (cf. [CF96], Chapters 14, 15 for explicit examples). In both situations we have $\text{Hom}(J_1, J_2) = 0$, which implies that there is no non-zero divisor class in $H^1(C_1) \otimes H^1(C_2)$. This completes the proof for $g = 4$.

2.12. *General case: end of proof.* — When $g > 4$, we may attach to both families a constant curve C_0 of genus $g - 4$ via a fixed point $c' \in C_0$, and use another fixed point $c'' \in C_0$ as the marked point.



We repeat the same procedure, and the proof is exactly the same. \square

2.13. *Corollary.* — When the base field k is uncountable (e.g. $k = \mathbb{C}$) and when $g \geq 4$, the same statement as in Theorem 2.6 holds for the Jacobian of a very general curve (over $\mathcal{M}_{g,1}$).

Proof. This is a consequence of the following fact. Let X, S be smooth connected quasi-projective varieties over k and $\pi: X \rightarrow S$ be a smooth projective map. Consider a cycle $\alpha \in \text{CH}(X)$. Via an argument using relative Hilbert schemes, one can show that the locus $\{s \in S : \alpha_s = 0 \in \text{CH}(X_s)\}$ is a countable union of Zariski-closed subset of S (cf. [Voi12], 2.1). Therefore, if k is uncountable and α is non-zero on the generic fiber, then α is non-zero on a very general fiber. \square

REFERENCES

- [Bea86] A. Beauville, *Sur l'anneau de Chow d'une variété abélienne*. Math. Ann. 273 (1986), no. 4, 647–651.
- [CF96] J. Cassels, E. Flynn, *Prolegomena to a middlebrow arithmetic of curves of genus 2*. London Mathematical Society Lecture Note Series, 230. Cambridge University Press, Cambridge, 1996. xiv+219 pp.
- [Del68] P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*. Inst. Hautes Études Sci. Publ. Math. (1968), no. 35, 259–278.
- [DM91] C. Deninger, J. Murre, *Motivic decomposition of abelian schemes and the Fourier transform*. J. Reine Angew. Math. 422 (1991), 201–219.
- [Fak96] N. Fakhruddin, *Algebraic cycles on generic abelian varieties*. Compositio Math. 100 (1996), no. 1, 101–119.
- [GG03] M. Green, P. Griffiths, *An interesting 0-cycle*. Duke Math. J. 119 (2003), no. 2, 261–313.
- [Pol07] A. Polishchuk, *Lie symmetries of the Chow group of a Jacobian and the tautological subring*. J. Algebraic Geom. 16 (2007), no. 3, 459–476.
- [Pol07b] A. Polishchuk, *Algebraic cycles on the relative symmetric powers and on the relative Jacobian of a family of curves. I*. Selecta Math. (N.S.) 13 (2007), no. 3, 531–569.
- [Roï80] A. Roïtman, *The torsion of the group of 0-cycles modulo rational equivalence*. Ann. of Math. (2) 111 (1980), no. 3, 553–569.
- [Voi12] C. Voisin, *Chow rings, decomposition of the diagonal and the topology of families*. Lecture notes (2012). <http://www.math.polytechnique.fr/~voisin/Articlesweb/weyllectures.pdf>
- [Yin13] Q. Yin, *Tautological cycles on generic Jacobians*. In preparation.

UNIVERSITÉ PIERRE ET MARIE CURIE PARIS VI, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE
 UNIVERSITEIT VAN AMSTERDAM, KORTEWEG-DE VRIES INSTITUUT VOOR WISKUNDE, POSTBUS 94248, 1090 GE
 AMSTERDAM, THE NETHERLANDS
 E-mail address: qyin@uva.nl